ON THE RATIONAL APPROXIMATION OF THE SUM OF THE RECIPROCALS OF THE FERMAT NUMBERS

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ABSTRACT. Let $\mathcal{G}(z) := \sum_{n=0}^{\infty} z^{2^n}(1 - z^{2^n})^{-1}$ denote the generating function of the ruler function, and $\mathcal{F}(z) := \sum_{n=0}^{\infty} z^{2^n}(1+z^{2^n})^{-1}$; note that the special value $\mathcal{F}(1/2)$ is the sum of the reciprocals of the Fermat numbers $F_n := 2^{2^n} + 1$. The functions $\mathcal{F}(z)$ and $\mathcal{G}(z)$ as well as their special values have been studied by Mahler, Golomb, Schwarz, and Duverney; it is known that the numbers $\mathcal{F}(\alpha)$ and $\mathcal{G}(\alpha)$ are transcendental for all algebraic numbers α which satisfy $0 < \alpha < 1$.

For a sequence $\mathbf{u}$, denote the Hankel matrix $H_n^p(\mathbf{u}) := (u(p+i+j-2))_{1\leqslant i,j\leqslant n}$. Let α be a real number. The *irrationality exponent* $\mu(\alpha)$ is defined as the supremum of the set of real numbers μ such that the inequality $|\alpha - p/q| < q^{-\mu}$ has infinitely many solutions $(p,q) \in \mathbb{Z} \times \mathbb{N}$.

In this paper, we first prove that the determinants of $H_n^1(\mathbf{g})$ and $H_n^1(\mathbf{f})$ are nonzero for every $n \geqslant 1$. We then use this result to prove that for $b \geqslant 2$ the irrationality exponents $\mu(\mathcal{F}(1/b))$ and $\mu(\mathcal{G}(1/b))$ are equal to 2; in particular, the irrationality exponent of the sum of the reciprocals of the Fermat numbers is 2.

1. Introduction

For $n \geqslant 0$ the nth *Fermat number* is given by $F_n := 2^{2^n} + 1$. In 1963, Golomb [Gol] proved that the sum of the reciprocals of the Fermat numbers is irrational and then in 2001 Duverney [Duv] proved transcendence though these results were probably known to Mahler as early as the late 1920s [Mah1, Mah2, Mah3]. In the same paper, Golomb proved something substantially more general; he defined the functions

$$(1)\qquad \mathcal{F}(z) := \sum_{n\geqslant 1} f(n)z^n = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1+z^{2^n}} \quad \text{and} \quad \mathcal{G}(z) := \sum_{n\geqslant 1} g(n)z^n = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1-z^{2^n}},$$

and showed that both $\mathcal{F}(1/b)$ and $\mathcal{G}(1/b)$ are irrational for all positive integers $b \geqslant 2$; note that the special value $\mathcal{F}(1/2)$ corresponds to the sum of the reciprocals of the Fermat numbers. Indeed, it is known that for $b \geqslant 2$ all of $\mathcal{F}(1/b)$ and $\mathcal{G}(1/b)$ are transcendental (this follows from results of Mahler [Mah1, Mah2, Mah3]; see Schwarz [Sch] and Coons [Coo] for details).

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Let α be a real number. The *irrationality exponent* $\mu(\alpha)$ is defined as the supremum of the set of real numbers μ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions $(p,q) \in \mathbb{Z} \times \mathbb{N}$. For example, Liouville [Lio] proved that for any sequence $\mathbf{a} := \{a(n)\}_{n\geqslant 0}$ with $a(n) \in \{0,1\}$ for all n and $a(n)$ not eventually zero, the numbers $\alpha(\mathbf{a}) := \sum_{n\geqslant 0} a(n)10^{-n!}$ have $\mu(\alpha(\mathbf{a})) = \infty$, and Roth [Rot] showed that if α is an irrational algebraic number, then $\mu(\alpha) = 2$. Note also that $\mu(\alpha) \geqslant 2$ for all irrational α.

In this paper, considering special values of the above series, we prove the following result.

Theorem 1. *Let $b \geqslant 2$ be a positive integer. Then $\mu(\mathcal{G}(1/b)) = \mu(\mathcal{F}(1/b)) = 2$. In particular,*

$$\mu\left(\sum_{n\geqslant 0} \frac{1}{2^{2^n}+1}\right) = 2.$$

Our method of proof is based on a method used recently by Bugeaud to prove that the irrationality exponent of the Thue–Morse–Mahler number is 2. To formalize, the Thue–Morse–Mahler sequence $\mathbf{t} := \{t(n)\}_{n\geqslant 0}$ is defined by $t(0) = 0$ and for $k \geqslant 0$, $t(2k) = t(k)$ and $t(2k+1) = 1 - t(k)$, and denote by $\mathcal{T}(z)$ the generating function

$$\mathcal{T}(z) = \sum_{k\geqslant 0} t(k) z^k.$$

Bugeaud [Bug] proved for every $b \geqslant 2$ that $\mu(\mathcal{T}(1/b)) = 2$. To do this, Bugeaud exploited a link between Padé approximants and Hankel matrices (this connection is recorded as Lemma 10 of Section 3 of this paper) combined with a result of Allouche, Peyriére, Wen and Wen [APWW, Theorem 2.1], to provide a good rational approximation to the generating function $\mathcal{T}(z)$, which was in turn used to prove his result.

For a sequence $\mathbf{u} = \{u(j)\}_{j\geqslant 0}$, we define the *Hankel matrix*

$$H_n^p(\mathbf{u}) := (u(p+i+j-2))_{1\leqslant i,j\leqslant n}.$$

The outline of this paper is as follows. In Section 2 we prove

Theorem 2. *Let $\mathbf{g} := \{g(n)\}_{n\geqslant 1}$ and $\mathbf{f} := \{f(n)\}_{n\geqslant 1}$ be the sequences defined in* (1). *The determinants of the Hankel matrices $H_n^1(\mathbf{g})$ and $H_n^1(\mathbf{f})$ are all nonzero.*

In Section 3 we use this result, via a link with Padé approximants, to prove Theorem 1.

2. Hankel determinants and the ruler function

Note that if $\mathbf{g} = \{g(n)\}_{n\geqslant 1}$ is the sequence given in (1), then $g(n)$ is equal to the 2–adic valuation of $2n$, known sometimes as the *ruler function*. For $n \geqslant 1$, the function $g(n)$ satisfies the recurrences

$$g(2n+1) = 1 \qquad \text{and} \qquad g(2n) = 1 + g(n).$$

This sequence starts

$$\mathbf{g} := \{g(n)\}_{n\geqslant 1} = \{1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,\ldots\}.$$

Since we will be working modulo 2, we have two choices for $g(0)$, and we will have to make both; thus, let $\mathbf{g}^0 := \{0, g(1), g(2), \cdots\}$ be the sequence starting at 0, and $\mathbf{g}^1 := \{1, g(1), g(2), \cdots\}$ be the sequence starting at 1.

We will need the following definitions and lemmas of Allouche, Peyrière, Wen and Wen [APWW]. The matrix $\mathbf{1}_{m\times n}$ is the $m\times n$ matrix with all its entries equal to 1, and $\mathbf{0}_{m\times n}$ is the $m\times n$ matrix with all its entries equal to 0. For the $n\times n$ square matrix A, we write $|A|$ and A^t for the determinant of A and the transpose of A, respectively, $\overline{A}$ for the matrix defined by

$$\overline{A} := \begin{pmatrix} A & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{n\times 1} & 0 \end{pmatrix},$$

and $A^{(j)}$ for the $n\times(n-1)$ matrix obtained by deleting the jth column of A. We write I_n for the $n\times n$ identity matrix, and

$$P_1(n) = (e_1, e_3, \ldots, e_{2\lfloor\frac{n+1}{2}\rfloor-1}, e_2, e_4, \ldots, e_{2\lfloor\frac{n}{2}\rfloor}),$$

where e_j is the column vector of length n with a 1 in its jth entry and zeros in the other entries. For convenience, throughout this paper we will write "$\equiv$" for equivalence modulo 2.

The main result of this section is the following theorem.

Theorem 3. *For all $n \geqslant 1$ we have*

$$|H_n^0(\mathbf{g}^0)| \equiv |H_n^2(\mathbf{g}^0)| \equiv |H_n^2(\mathbf{g}^1)| \equiv \begin{cases} 0 & \text{if } n\equiv 1,4 \pmod 6 \\ 1 & \text{if } n \equiv 0,2,3,5 \pmod 6, \end{cases}$$

$$|\overline{H_n^0(\mathbf{g}^0)}| \equiv \begin{cases} 0 & \text{if } n\equiv 2,3 \pmod 6 \\ 1 & \text{if } n \equiv 0,1,4,5 \pmod 6, \end{cases}$$

$$|H_n^0(\mathbf{g}^1)| \equiv \begin{cases} 0 & \text{if } n\equiv 1,2,4,5 \pmod 6 \\ 1 & \text{if } n \equiv 0,3 \pmod 6, \end{cases}$$

$$|\overline{H_n^0(\mathbf{g}^1)}| \equiv \begin{cases} 0 & \text{if } n\equiv 0,1,2,3 \pmod 6 \\ 1 & \text{if } n \equiv 4,5 \pmod 6, \end{cases}$$

$$|H_n^1(\mathbf{g}^0)| \equiv |H_n^1(\mathbf{g}^1)| \equiv 1,$$

$$|\overline{H_n^1(\mathbf{g}^0)}| \equiv |\overline{H_n^1(\mathbf{g}^1)}| \equiv \begin{cases} 0 & \text{if } n\equiv 0,2,4 \pmod 6 \\ 1 & \text{if } n \equiv 1,3,5 \pmod 6, \end{cases}$$

$$|H_n^2(\mathbf{g}^0)| \equiv |H_n^2(\mathbf{g}^1)| \equiv \begin{cases} 0 & \text{if } n\equiv 1,4 \pmod 6 \\ 1 & \text{if } n \equiv 0,2,3,5 \pmod 6, \end{cases}$$

$$|\overline{H_n^2(\mathbf{g}^0)}| \equiv |\overline{H_n^2(\mathbf{g}^1)}| \equiv \begin{cases} 0 & \text{if } n\equiv 0,5 \pmod 6 \\ 1 & \text{if } n \equiv 1,2,3,4 \pmod 6. \end{cases}$$

To prove Theorem 3 we will rely heavily in the following three lemmas, which originally occurred as Lemmas 1.2, 1.3, and 1.4 of [APWW].

Lemma 4 ((Allouche et al. [APWW])). *Let A and B be two square matrices of order m and n respectively, and a, b, x and y for numbers. One has*

$$\begin{vmatrix} aA & y\mathbf{1}_{m\times n} \\ x\mathbf{1}_{n\times m} & bB \end{vmatrix} = a^m b^n |A|\cdot|B| - xya^{m-1}b^{m-1}|\overline{A}|\cdot|\overline{B}|.$$

Lemma 5 ((Allouche et al. [APWW])). *Let A, B, and C be three square matrices of order m, n, and p respectively, and three numbers a, b, and c. One has*

$$\begin{vmatrix} A & c\mathbf{1}_{m\times n} & b\mathbf{1}_{m\times p} \\ c\mathbf{1}_{n\times m} & B & a\mathbf{1}_{n\times p} \\ b\mathbf{1}_{p\times m} & a\mathbf{1}_{p\times n} & C \end{vmatrix} = |A|\cdot|B|\cdot|C| - a^2|A|\cdot|\overline{B}|\cdot|\overline{C}| - b^2|\overline{A}|\cdot|B|\cdot|\overline{C}|$$
$$- c^2|\overline{A}|\cdot|\overline{B}|\cdot|C| - 2abc|\overline{A}|\cdot|\overline{B}|\cdot|\overline{C}|.$$

Lemma 6 ((Allouche et al. [APWW])). *Let $x\in\mathbb{R}$ and A be an $m\times m$ matrix, then*

(i) $|x\mathbf{1}_{m\times m} + A| = |A| - x|\overline{A}|$,

(ii) $|\overline{x\mathbf{1}_{m\times m} + A}| = |\overline{A}|$,

(iii) $|\overline{-A}| = (-1)^{m+1}|\overline{A}|$.

For a sequence $\mathbf{u} = \{u(j)\}_{j\geqslant 0}$ define the matrix $K_n^p(\mathbf{u})$ by

$$K_n^p(\mathbf{u}) := (u(p+2(i+j-2)))_{1\leqslant i,j\leqslant n}.$$

Lemma 7. *For all $n\geqslant 1$, we have*

(i') $|H_{2n}^0(\mathbf{g}^1)| = |H_n^0(\mathbf{g}^0)|\cdot|H_n^1(\mathbf{g}^1)| - |\overline{H_n^0(\mathbf{g}^0)}|\cdot|H_n^1(\mathbf{g}^1)| - |H_n^0(\mathbf{g}^0)|\cdot|\overline{H_n^1(\mathbf{g}^1)}|$,
$|H_{2n}^0(\mathbf{g}^0)| = |H_n^0(\mathbf{g}^1)|\cdot|H_n^1(\mathbf{g}^1)| - |\overline{H_n^0(\mathbf{g}^1)}|\cdot|H_n^1(\mathbf{g}^1)| - |H_n^0(\mathbf{g}^1)|\cdot|\overline{H_n^1(\mathbf{g}^1)}|$,

(i”) *for $p\geqslant 1$,*

$$|H_{2n}^{2p}(\mathbf{g}^1)| = |H_n^p(\mathbf{g}^1)|\cdot|H_n^{p+1}(\mathbf{g}^1)| - |\overline{H_n^p(\mathbf{g}^1)}|\cdot|H_n^{p+1}(\mathbf{g}^1)|$$
$$- |H_n^p(\mathbf{g}^1)|\cdot|\overline{H_n^{p+1}(\mathbf{g}^1)}|,$$

(ii') $|\overline{H_{2n}^0(\mathbf{g}^1)}| \equiv |H_n^0(\mathbf{g}^0)|\cdot|\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_n^0(\mathbf{g}^0)}|\cdot|H_n^1(\mathbf{g}^1)|$,
$|\overline{H_{2n}^0(\mathbf{g}^0)}| \equiv |H_n^0(\mathbf{g}^1)|\cdot|\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_n^0(\mathbf{g}^1)}|\cdot|H_n^1(\mathbf{g}^1)|$,

(ii”) *for $p\geqslant 1$,* $|\overline{H_{2n}^{2p}(\mathbf{g}^1)}| \equiv |H_n^p(\mathbf{g}^1)|\cdot|\overline{H_n^{p+1}(\mathbf{g}^1)}| + |\overline{H_n^p(\mathbf{g}^1)}|\cdot|H_n^{p+1}(\mathbf{g}^1)|$

(iii') $|H_{2n+1}^0(\mathbf{g}^1)| = |H_{n+1}^0(\mathbf{g}^0)|\cdot|H_n^1(\mathbf{g}^1)| - |\overline{H_{n+1}^0(\mathbf{g}^0)}|\cdot|H_n^1(\mathbf{g}^1)|$
$-|H_{n+1}^0(\mathbf{g}^0)|\cdot|\overline{H_n^1(\mathbf{g}^1)}|$
$|H_{2n+1}^0(\mathbf{g}^0)| = |H_{n+1}^0(\mathbf{g}^1)|\cdot|H_n^1(\mathbf{g}^1)| - |\overline{H_{n+1}^0(\mathbf{g}^1)}|\cdot|H_n^1(\mathbf{g}^1)|$
$-|H_{n+1}^0(\mathbf{g}^1)|\cdot|\overline{H_n^1(\mathbf{g}^1)}|$,

(iii”) *for $p\geqslant 1$,*

$$|H_{2n+1}^{2p}(\mathbf{g}^1)| = |H_{n+1}^p(\mathbf{g}^1)|\cdot|H_n^{p+1}(\mathbf{g}^1)| - |\overline{H_{n+1}^p(\mathbf{g}^1)}|\cdot|H_n^{p+1}(\mathbf{g}^1)|$$
$$- |H_{n+1}^p(\mathbf{g}^1)|\cdot|\overline{H_n^{p+1}(\mathbf{g}^1)}|,$$

(iv') $|\overline{H_{2n+1}^0(\mathbf{g}^1)}| \equiv |H_{n+1}^0(\mathbf{g}^0)|\cdot|\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_{n+1}^0(\mathbf{g}^0)}|\cdot|H_n^1(\mathbf{g}^1)|$,
$|\overline{H_{2n+1}^0(\mathbf{g}^0)}| \equiv |H_{n+1}^0(\mathbf{g}^1)|\cdot|\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_{n+1}^0(\mathbf{g}^1)}|\cdot|H_n^1(\mathbf{g}^1)|$,

(iv") *for* $p \geqslant 1$,

$$|\overline{H^{2p}_{2n+1}(\mathbf{g}^1)}| \equiv |H^{p}_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^{p+1}_{n}(\mathbf{g}^1)}| + |\overline{H^{p}_{n+1}(\mathbf{g}^1)}| \cdot |H^{p+1}_{n}(\mathbf{g}^1)|,$$

(v) *for* $p \geqslant 0$, $|H^{2p+1}_{2n}(\mathbf{g}^1)| \equiv |H^{p+1}_{n}(\mathbf{g}^1)|$,

(vi) *for* $p \geqslant 0$, $|\overline{H^{2p+1}_{2n}(\mathbf{g}^1)}| \equiv 0$,

(vii')
$$\begin{aligned}|H^1_{2n+1}(\mathbf{g}^1)| \equiv \Big[\Big(&|H^0_{n+1}(\mathbf{g}^0)| \cdot |H^1_n(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^0)}| \cdot |H^1_n(\mathbf{g}^1)| \\ &-|H^0_{n+1}(\mathbf{g}^0)| \cdot |\overline{H^1_n(\mathbf{g}^1)}|\Big) \\ \times\Big(&|H^1_{n+1}(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_{n+1}(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| - |H^1_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}|\Big)\Big] \\ &- \Big[\Big(|H^1_n(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_n(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| - |H^1_n(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}|\Big) \\ \times\Big(&|H^0_{n+1}(\mathbf{g}^0)| \cdot |H^1_{n+1}(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^0)}| \cdot |H^1_{n+1}(\mathbf{g}^1)| \\ &- |H^0_{n+1}(\mathbf{g}^0)| \cdot |\overline{H^1_n(\mathbf{g}^1)}|\Big)\Big],\end{aligned}$$

$$\begin{aligned}|H^1_{2n+1}(\mathbf{g}^0)| \equiv \Big[\Big(&|H^0_{n+1}(\mathbf{g}^1)| \cdot |H^1_n(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^1)}| \cdot |H^1_n(\mathbf{g}^1)| \\ &-|H^0_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^1_{n+1}(\mathbf{g}^1)}|\Big) \\ \times\Big(&|H^1_{n+1}(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_{n+1}(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| - |H^1_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}|\Big)\Big] \\ &- \Big[\Big(|H^1_n(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_n(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| - |H^1_n(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}|\Big) \\ \times\Big(&|H^0_{n+1}(\mathbf{g}^1)| \cdot |H^1_{n+1}(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^1)}| \cdot |H^1_{n+1}(\mathbf{g}^1)| \\ &- |H^0_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^1_{n+1}(\mathbf{g}^1)}|\Big)\Big],\end{aligned}$$

(vii") *for* $p \geqslant 1$,

$$\begin{aligned}|H^{2p+1}_{2n+1}(\mathbf{g}^1)| \equiv &\Big(|H^{p+2}_n(\mathbf{g}^1)| \cdot |\overline{H^{p}_{n+1}(\mathbf{g}^1)}| - |H^{p}_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^{p+2}_n(\mathbf{g}^1)}|\Big) \\ &\times \Big(|H^{p+1}_n(\mathbf{g}^1)| \cdot |\overline{H^{p+1}_{n+1}(\mathbf{g}^1)}| - |H^{p+1}_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^{p+1}_n(\mathbf{g}^1)}|\Big),\end{aligned}$$

(viii')
$$\begin{aligned}|\overline{H^1_{2n+1}(\mathbf{g}^1)}| \equiv &\Big(|H^2_n(\mathbf{g}^1)| \cdot |\overline{H^0_{n+1}(\mathbf{g}^1)}| - |H^0_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}|\Big) \\ &\times \Big(|H^1_n(\mathbf{g}^1)| \cdot |\overline{H^1_{n+1}(\mathbf{g}^1)}| - |H^1_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^1_n(\mathbf{g}^1)}|\Big) \\ &+ |H^0_{n+1}(\mathbf{g}^0)| \cdot |H^1_n(\mathbf{g}^1)| \cdot |H^1_{n+1}(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| \\ &+ |H^1_n(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| \cdot |H^0_{n+2}(\mathbf{g}^0)| \cdot |\overline{H^1_{n+2}(\mathbf{g}^1)}|,\end{aligned}$$

(viii") *for* $p \geqslant 1$

$$\begin{aligned}|\overline{H^{2p+1}_{2n+1}(\mathbf{g}^1)}| \equiv \Big(&|H^{p+2}_n(\mathbf{g}^1)|\cdot|\overline{H^{p}_{n+1}(\mathbf{g}^1)}| - |H^{p}_{n+1}(\mathbf{g}^1)|\cdot|\overline{H^{p+2}_{n}(\mathbf{g}^1)}|\Big)\\ &\times\Big(|H^{p+1}_n(\mathbf{g}^1)|\cdot|\overline{H^{p+1}_{n+1}(\mathbf{g}^1)}| - |H^{p+1}_{n+1}(\mathbf{g}^1)|\cdot|\overline{H^{p+1}_{n}(\mathbf{g}^1)}|\Big)\\ &+|H^{p}_{n+1}(\mathbf{g}^1)|\cdot|H^{p+1}_{n}(\mathbf{g}^1)|\cdot|H^{p+1}_{n+1}(\mathbf{g}^1)|\cdot|H^{p+2}_{n}(\mathbf{g}^1)|\\ &+|H^{p+1}_{n}(\mathbf{g}^1)|\cdot|H^{p+2}_{n}(\mathbf{g}^1)|\cdot|H^{p}_{n+2}(\mathbf{g}^1)|\cdot|H^{p+1}_{n+2}(\mathbf{g}^1)|.\end{aligned}$$

Proof. For $p \geqslant 1$ we have that

$$\begin{aligned}(2)\quad K^{2p}_n(\mathbf{g}^0) = K^{2p}_n(\mathbf{g}^1) &= (g(2p+2(i+j-2)))_{1\leqslant i,j\leqslant n}\\ &= (1+g(p+(i+j-2)))_{1\leqslant i,j\leqslant n} = \mathbf{1}_{n\times n} + H^p_n(\mathbf{g}^1),\end{aligned}$$

and for $p \geqslant 0$ that

$$\begin{aligned}(3)\quad K^{2p+1}_n(\mathbf{g}^1) &= (g(2p+1+2(i+j-2)))_{1\leqslant i,j\leqslant n}\\ &= (g(2(p+i+j-2)+1))_{1\leqslant i,j\leqslant n} = \mathbf{1}_{n\times n}.\end{aligned}$$

The analogue of (2) for $p = 0$ must take into account the difference of $\mathbf{g}^1$ and $\mathbf{g}^0$ in there first coordinate. Since $g^0(0) \equiv 1 + g^1(0) \pmod 2$ and $g^1(0) \equiv 1 + g^0(0) \pmod 2$, we have that

$$(4)\qquad K^0_n(\mathbf{g}^0) = \mathbf{1}_{n\times n} + H^0_n(\mathbf{g}^1), \qquad \text{and} \qquad K^0_n(\mathbf{g}^1) = \mathbf{1}_{n\times n} + H^0_n(\mathbf{g}^0).$$

Note that for P_1 defined above we have (see equation (8) of [APWW]) that

$$(5)\qquad P_1^t H^p_{2n}(\mathbf{u}) P_1 = \begin{pmatrix} K^p_n(\mathbf{u}) & K^{p+1}_n(\mathbf{u}) \\ K^{p+1}_n(\mathbf{u}) & K^{p+2}_n(\mathbf{u})\end{pmatrix},$$

and

$$(6)\qquad P_1^t H^p_{2n+1}(\mathbf{u}) P_1 = \begin{pmatrix} K^p_{n+1}(\mathbf{u}) & (K^{p+1}_{n+1}(\mathbf{u}))^{(n+1)} \\ (K^{p+1}_{n+1}(\mathbf{u}))^{(n+1)t} & K^{p+2}_n(\mathbf{u})\end{pmatrix}.$$

To prove (i) we must now break into two cases. If $p = 0$, then using (5), (2) and (3), we have that

$$\begin{aligned}P_1^t H^0_{2n}(\mathbf{g}^1) P_1 &= \begin{pmatrix} K^0_n(\mathbf{g}^1) & K^1_n(\mathbf{g}^1) \\ K^1_n(\mathbf{g}^1) & K^2_n(\mathbf{g}^1)\end{pmatrix}\\ &= \begin{pmatrix} \mathbf{1}_{n\times n} + H^0_n(\mathbf{g}^0) & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} + H^1_n(\mathbf{g}^1)\end{pmatrix},\end{aligned}$$

so that Lemma 4 gives

$$\begin{aligned}|H^0_{2n}(\mathbf{g}^1)| &= |\mathbf{1}_{n\times n} + H^0_n(\mathbf{g}^0)|\cdot|\mathbf{1}_{n\times n} + H^1_n(\mathbf{g}^1)|\\ &\quad - |\overline{\mathbf{1}_{n\times n} + H^0_n(\mathbf{g}^0)}|\cdot|\overline{\mathbf{1}_{n\times n} + H^1_n(\mathbf{g}^1)}|\\ &= |H^0_n(\mathbf{g}^0)|\cdot|H^1_n(\mathbf{g}^1)| - |\overline{H^0_n(\mathbf{g}^0)}|\cdot|H^1_n(\mathbf{g}^1)| - |H^0_n(\mathbf{g}^0)|\cdot|\overline{H^1_n(\mathbf{g}^1)}|\end{aligned}$$

The similar result holds for $|H^0_{2n}(\mathbf{g}^0)|$ by replacing $\mathbf{g}^1$ with $\mathbf{g}^0$ in the above argument. This proves (i').

If $p \geqslant 1$, then again that using (5), (2) and (3), we have that

$$P_1^t H_{2n}^{2p}(\mathbf{g}^1) P_1 = \begin{pmatrix} K_n^{2p}(\mathbf{g}^1) & K_n^{2p+1}(\mathbf{g}^1) \\ K_n^{2p+1}(\mathbf{g}^1) & K_n^{2p+2}(\mathbf{g}^1) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1}_{n\times n} + H_n^p(\mathbf{g}^1) & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) \end{pmatrix},$$

so that Lemma 4 gives

$$\begin{aligned} |H_{2n}^{2p}(\mathbf{g}^1)| &= |\mathbf{1}_{n\times n} + H_n^p(\mathbf{g}^1)| \cdot |\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)| \\ &\quad - |\overline{\mathbf{1}_{n\times n} + H_n^p(\mathbf{g}^1)}| \cdot |\overline{\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)}| \\ &= |H_n^p(\mathbf{g}^1)| \cdot |H_n^{p+1}(\mathbf{g}^1)| - |\overline{H_n^p(\mathbf{g}^1)}| \cdot |H_n^{p+1}(\mathbf{g}^1)| - |H_n^p(\mathbf{g}^1)| \cdot |\overline{H_n^{p+1}(\mathbf{g}^1)}|, \end{aligned}$$

which proves (i").

For (ii'), we have that

$$\begin{aligned} &\begin{pmatrix} P_1^t & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \overline{H_{2n}^0(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} P_1^t & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \begin{pmatrix} H_{2n}^0(\mathbf{g}^1) & \mathbf{1}_{2n\times 1} \\ \mathbf{1}_{1\times 2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} P_1^t H_{2n}^0(\mathbf{g}^1) P_1 & P_1^t \mathbf{1}_{2n\times 1} \\ \mathbf{1}_{1\times 2n} P_1 & 0 \end{pmatrix} \\ &\qquad = \begin{pmatrix} \mathbf{1}_{n\times n} + H_n^0(\mathbf{g}^0) & \mathbf{1}_{n\times n} & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1) & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{1\times n} & \mathbf{1}_{1\times n} & 0 \end{pmatrix}. \end{aligned}$$

If we consider the 1×1 matrix (0), then

$$\overline{(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that $|(0)| = 0$ and $|\overline{(0)}| = -1$. Thus Lemma 5 gives

$$\begin{aligned} |\overline{H_{2n}^0(\mathbf{g}^1)}| &= |\mathbf{1}_{n\times n} + H_n^0(\mathbf{g}^0)| \cdot |\overline{\mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1)}| \\ &\quad + |\overline{\mathbf{1}_{n\times n} + H_n^0(\mathbf{g}^0)}| \cdot |\mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1)| \\ &\quad + 2|\overline{\mathbf{1}_{n\times n} + H_n^0(\mathbf{g}^0)}| \cdot |\overline{\mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1)}| \\ &= \Big(|H_n^0(\mathbf{g}^0)| - |\overline{H_n^0(\mathbf{g}^0)}|\Big) \cdot |\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_n^0(\mathbf{g}^0)}| \cdot \Big(|H_n^1(\mathbf{g}^1)| - |\overline{H_n^1(\mathbf{g}^1)}|\Big) \\ &\quad + 2|\overline{H_n^0(\mathbf{g}^0)}| \cdot |\overline{H_n^1(\mathbf{g}^1)}| \\ &\equiv |H_n^0(\mathbf{g}^0)| \cdot |\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_n^0(\mathbf{g}^0)}| \cdot |H_n^1(\mathbf{g}^1)|. \end{aligned}$$

The similar result holds for $|\overline{H_{2n}^0(\mathbf{g}^0)}|$ by replacing $\mathbf{g}^1$ with $\mathbf{g}^0$ in the above argument. This proves (ii').

For (ii"), note that for $p \geqslant 1$ we have

$$\begin{aligned}
&\begin{pmatrix} P_1^t & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \overline{H_{2n}^{2p}(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} P_1^t & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \begin{pmatrix} H_{2n}^{2p}(\mathbf{g}^1) & \mathbf{1}_{2n\times 1} \\ \mathbf{1}_{1\times 2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} P_1^t H_{2n}^{2p}(\mathbf{g}^1) P_1 & P_1^t \mathbf{1}_{2n\times 1} \\ \mathbf{1}_{1\times 2n} P_1 & 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} \mathbf{1}_{n\times n} + H_n^p(\mathbf{g}^1) & \mathbf{1}_{n\times n} & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{1\times n} & \mathbf{1}_{1\times n} & 0 \end{pmatrix}.
\end{aligned}$$

Using the above comments about $|(0)|$ and $|\overline{(0)}|$ and Lemma 5 gives

$$\begin{aligned}
|\overline{H_{2n}^{2p}(\mathbf{g}^1)}| &= |\mathbf{1}_{n\times n} + H_n^p(\mathbf{g}^1)| \cdot |\overline{\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)}| \\
&\qquad + |\overline{\mathbf{1}_{n\times n} + H_n^p(\mathbf{g}^1)}| \cdot |\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)| \\
&\qquad\qquad + 2|\overline{\mathbf{1}_{n\times n} + H_n^p(\mathbf{g}^1)}| \cdot |\overline{\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)}| \\
&= \left(|H_n^p(\mathbf{g}^1)| - |\overline{H_n^p(\mathbf{g}^1)}|\right) \cdot |\overline{H_n^{p+1}(\mathbf{g}^1)}| \\
&\qquad + |\overline{H_n^p(\mathbf{g}^1)}| \cdot \left(|H_n^{p+1}(\mathbf{g}^1)| - |\overline{H_n^{p+1}(\mathbf{g}^1)}|\right) \\
&\qquad + 2|\overline{H_n^p(\mathbf{g}^1)}| \cdot |\overline{H_n^{p+1}(\mathbf{g}^1)}| \\
&\equiv |H_n^p(\mathbf{g}^1)| \cdot |\overline{H_n^{p+1}(\mathbf{g}^1)}| + |\overline{H_n^p(\mathbf{g}^1)}| \cdot |H_n^{p+1}(\mathbf{g}^1)|.
\end{aligned}$$

For (iii'), note that

$$\begin{aligned}
P_1^t H_{2n+1}^0(\mathbf{g}^1) P_1 &= \begin{pmatrix} K_{n+1}^0(\mathbf{g}^1) & (K_{n+1}^1(\mathbf{g}^1))^{(n+1)} \\ (K_{n+1}^1(\mathbf{g}^1))^{(n+1)t} & K_n^2(\mathbf{g}^1) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_{(n+1)\times(n+1)} + H_{n+1}^0(\mathbf{g}^0) & (\mathbf{1}_{(n+1)\times(n+1)})^{(n+1)} \\ (\mathbf{1}_{(n+1)\times(n+1)})^{(n+1)t} & \mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_{(n+1)\times(n+1)} + H_{n+1}^0(\mathbf{g}^0) & \mathbf{1}_{(n+1)\times n} \\ \mathbf{1}_{n\times(n+1)} & \mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1) \end{pmatrix}. \qquad (7)
\end{aligned}$$

Thus we have

$$\begin{aligned}
|H_{2n+1}^0(\mathbf{g}^1)| &= |\mathbf{1}_{(n+1)\times(n+1)} + H_{n+1}^0(\mathbf{g}^0)| \cdot |\mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1)| \\
&\qquad - |\overline{\mathbf{1}_{(n+1)\times(n+1)} + H_{n+1}^0(\mathbf{g}^0)}| \cdot |\overline{\mathbf{1}_{n\times n} + H_n^1(\mathbf{g}^1)}| \\
&= \left(|H_{n+1}^0(\mathbf{g}^0)| - |\overline{H_{n+1}^0(\mathbf{g}^0)}|\right) \cdot \left(|H_n^1(\mathbf{g}^1)| - |\overline{H_n^1(\mathbf{g}^1)}|\right) \\
&\qquad - |\overline{H_{n+1}^0(\mathbf{g}^0)}| \cdot |\overline{H_n^1(\mathbf{g}^1)}| \\
&= |H_{n+1}^0(\mathbf{g}^0)| \cdot |H_n^1(\mathbf{g}^1)| - |\overline{H_{n+1}^0(\mathbf{g}^0)}| \cdot |H_n^1(\mathbf{g}^1)| \\
&\qquad - |H_{n+1}^0(\mathbf{g}^0)| \cdot |\overline{H_n^1(\mathbf{g}^1)}|.
\end{aligned}$$

The similar result holds for $|\overline{H^0_{2n}(\mathbf{g}^0)}|$ by replacing $\mathbf{g}^1$ with $\mathbf{g}^0$ in the above argument. This proves (iii').

For (iii"), $p \geqslant 1$ so that

$$
\begin{aligned}
P_1^t H^{2p}_{2n+1}(\mathbf{g}^1) P_1 &= \begin{pmatrix} K^{2p}_{n+1}(\mathbf{g}^1) & (K^{2p+1}_{n+1}(\mathbf{g}^1))^{(n+1)} \\ (K^{2p+1}_{n+1}(\mathbf{g}^1))^{(n+1)t} & K^{2p+2}_{n}(\mathbf{g}^1) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1) & (\mathbf{1}_{(n+1)\times(n+1)})^{(n+1)} \\ (\mathbf{1}_{(n+1)\times(n+1)})^{(n+1)t} & \mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1) & \mathbf{1}_{(n+1)\times n} \\ \mathbf{1}_{n\times(n+1)} & \mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1) \end{pmatrix}. \qquad (8)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
|H^{2p}_{2n+1}(\mathbf{g}^1)| &= |\mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1)| \cdot |\mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1)| \\
&\qquad - |\overline{\mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1)}| \cdot |\overline{\mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1)}| \\
&= \Big(|H^p_{n+1}(\mathbf{g}^1)| - |\overline{H^p_{n+1}(\mathbf{g}^1)}|\Big) \cdot \Big(|H^{p+1}_n(\mathbf{g}^1)| - |\overline{H^{p+1}_n(\mathbf{g}^1)}|\Big) \\
&\qquad - |\overline{H^p_{n+1}(\mathbf{g}^1)}| \cdot |\overline{H^{p+1}_n(\mathbf{g}^1)}| \\
&= |H^p_{n+1}(\mathbf{g}^1)| \cdot |H^{p+1}_n(\mathbf{g}^1)| - |\overline{H^p_{n+1}(\mathbf{g}^1)}| \cdot |H^{p+1}_n(\mathbf{g}^1)| \\
&\qquad - |H^p_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^{p+1}_n(\mathbf{g}^1)}|.
\end{aligned}
$$

For (iv'), similar to (ii') we have

$$
\begin{aligned}
&\begin{pmatrix} P_1^t & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \overline{H^0_{2n+1}(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} P_1^t & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \begin{pmatrix} H^0_{2n+1}(\mathbf{g}^1) & \mathbf{1}_{(2n+1)\times 1} \\ \mathbf{1}_{1\times(2n+1)} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} P_1^t H^0_{2n+1}(\mathbf{g}^1) P_1 & P_1^t \mathbf{1}_{(2n+1)\times 1} \\ \mathbf{1}_{1\times(2n+1)} P_1 & 0 \end{pmatrix}.
\end{aligned}
$$

Thus using (7) we have

$$
\begin{aligned}
&\begin{pmatrix} P_1^t & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \overline{H^0_{2n+1}(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} \mathbf{1}_{(n+1)\times(n+1)} + H^0_{n+1}(\mathbf{g}^0) & \mathbf{1}_{(n+1)\times n} & \mathbf{1}_{(n+1)\times 1} \\ \mathbf{1}_{n\times(n+1)} & \mathbf{1}_{n\times n} + H^1_n(\mathbf{g}^1) & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{1\times(n+1)} & \mathbf{1}_{1\times n} & 0 \end{pmatrix}.
\end{aligned}
$$

Just as before, we have $|(0)|=0$ and $|\overline{(0)}|=-1$, and so again using Lemma 5 we have

$$\begin{aligned}
|\overline{H^0_{2n+1}(\mathbf{g}^1)}| &= |\mathbf{1}_{(n+1)\times(n+1)} + H^0_{n+1}(\mathbf{g}^0)| \cdot |\overline{\mathbf{1}_{n\times n} + H^1_n(\mathbf{g}^1)}| \\
&\qquad + |\overline{\mathbf{1}_{(n+1)\times(n+1)} + H^0_{n+1}(\mathbf{g}^0)}| \cdot |\mathbf{1}_{n\times n} + H^1_n(\mathbf{g}^1)| \\
&\qquad\quad + 2|\overline{\mathbf{1}_{(n+1)\times(n+1)} + H^0_{n+1}(\mathbf{g}^0)}| \cdot |\overline{\mathbf{1}_{n\times n} + H^1_n(\mathbf{g}^1)}| \\
&= \left(|H^0_{n+1}(\mathbf{g}^0)| - |\overline{H^0_{n+1}(\mathbf{g}^0)}|\right) \cdot |\overline{H^1_n(\mathbf{g}^1)}| \\
&\qquad + |\overline{H^0_{n+1}(\mathbf{g}^0)}| \cdot \left(|H^1_n(\mathbf{g}^1)| - |\overline{H^1_n(\mathbf{g}^1)}|\right) + 2|\overline{H^0_{n+1}(\mathbf{g}^0)}| \cdot |\overline{H^1_n(\mathbf{g}^1)}| \\
&\equiv |H^0_{n+1}(\mathbf{g}^0)| \cdot |\overline{H^1_n(\mathbf{g}^1)}| + |\overline{H^0_{n+1}(\mathbf{g}^0)}| \cdot |H^1_n(\mathbf{g}^1)|.
\end{aligned}$$

The similar result holds for $|\overline{H^0_{2n}(\mathbf{g}^0)}|$ by replacing $\mathbf{g}^1$ with $\mathbf{g}^0$ in the above argument. This proves (iv').

For (iv"), similar to (ii") we have

$$\begin{aligned}
&\begin{pmatrix} P_1^t & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \overline{H^{2p}_{2n+1}(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} P_1^t & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \begin{pmatrix} H^{2p}_{2n+1}(\mathbf{g}^1) & \mathbf{1}_{(2n+1)\times 1} \\ \mathbf{1}_{1\times(2n+1)} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} P_1^t H^{2p}_{2n+1}(\mathbf{g}^1) P_1 & P_1^t \mathbf{1}_{(2n+1)\times 1} \\ \mathbf{1}_{1\times(2n+1)} P_1 & 0 \end{pmatrix}.
\end{aligned}$$

Thus using (8) we have

$$\begin{aligned}
&\begin{pmatrix} P_1^t & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \overline{H^{2p}_{2n+1}(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{(2n+1)\times 1} \\ \mathbf{0}_{1\times(2n+1)} & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} \mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1) & \mathbf{1}_{(n+1)\times n} & \mathbf{1}_{(n+1)\times 1} \\ \mathbf{1}_{n\times(n+1)} & \mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1) & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{1\times(n+1)} & \mathbf{1}_{1\times n} & 0 \end{pmatrix}.
\end{aligned}$$

Just as before, we have $|(0)|=0$ and $|\overline{(0)}|=-1$, and so again by one of the above lemmas

$$\begin{aligned}
|\overline{H^{2p}_{2n+1}(\mathbf{g}^1)}| &= |\mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1)| \cdot |\overline{\mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1)}| \\
&\qquad + |\overline{\mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1)}| \cdot |\mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1)| \\
&\qquad\quad + 2|\overline{\mathbf{1}_{(n+1)\times(n+1)} + H^p_{n+1}(\mathbf{g}^1)}| \cdot |\overline{\mathbf{1}_{n\times n} + H^{p+1}_n(\mathbf{g}^1)}| \\
&= \left(|H^p_{n+1}(\mathbf{g}^1)| - |\overline{H^p_{n+1}(\mathbf{g}^1)}|\right) \cdot |\overline{H^{p+1}_n(\mathbf{g}^1)}| \\
&\qquad + |\overline{H^p_{n+1}(\mathbf{g}^1)}| \cdot \left(|H^{p+1}_n(\mathbf{g}^1)| - |\overline{H^{p+1}_n(\mathbf{g}^1)}|\right) \\
&\qquad\quad + 2|\overline{H^p_{n+1}(\mathbf{g}^1)}| \cdot |\overline{H^{p+1}_n(\mathbf{g}^1)}| \\
&\equiv |H^p_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^{p+1}_n(\mathbf{g}^1)}| + |\overline{H^p_{n+1}(\mathbf{g}^1)}| \cdot |H^{p+1}_n(\mathbf{g}^1)|.
\end{aligned}$$

For (v), we have

$$\begin{pmatrix} \mathbf{0}_{n\times n} & I_n \\ I_n & \mathbf{0}_{n\times n} \end{pmatrix} P_1^t H_{2n}^{2p+1}(\mathbf{g}^1) P_1 = \begin{pmatrix} K_n^{2p+2}(\mathbf{g}^1) & K_n^{2p+3}(\mathbf{g}^1) \\ K_n^{2p+1}(\mathbf{g}^1) & K_n^{2p+2}(\mathbf{g}^1) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) \end{pmatrix},$$

and so

$$\begin{aligned} |H_{2n}^{2p+1}(\mathbf{g}^1)| &= |\overline{\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)}|^2 - |\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)|^2 \\ &= |\overline{H_n^{p+1}(\mathbf{g}^1)}|^2 - \Big(|H_n^{p+1}(\mathbf{g}^1)| - |\overline{H_n^{p+1}(\mathbf{g}^1)}|\Big)^2 \\ &= -|H_n^{p+1}(\mathbf{g}^1)|^2 + 2|H_n^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_n^{p+1}(\mathbf{g}^1)}| \\ &\equiv |H_n^{p+1}(\mathbf{g}^1)|. \end{aligned}$$

For (vi), similar to (ii) we have using (5) that

$$\begin{aligned} &\begin{pmatrix} P_1^t & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \overline{H_{2n}^{2p+1}(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_1^t & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \begin{pmatrix} H_{2n}^{2p+1}(\mathbf{g}^1) & \mathbf{1}_{2n\times 1} \\ \mathbf{1}_{1\times 2n} & 0 \end{pmatrix} \begin{pmatrix} P_1 & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_1^t H_{2n}^{2p+1}(\mathbf{g}^1) P_1 & P_1^t \mathbf{1}_{2n\times 1} \\ \mathbf{1}_{1\times 2n} P_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) & \mathbf{1}_{n\times n} & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{1\times n} & \mathbf{1}_{1\times n} & 0 \end{pmatrix}. \end{aligned}$$

Thus we have that

$$\begin{pmatrix} \mathbf{0}_{n\times n} & I_n & \mathbf{0}_{n\times 1} \\ I_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 1 \end{pmatrix} \begin{pmatrix} P_1^t & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix} \overline{H_{2n}^{2p+1}(\mathbf{g}^1)} \begin{pmatrix} P_1 & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) & \mathbf{1}_{n\times n} & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1) & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{1\times n} & \mathbf{1}_{1\times n} & 0 \end{pmatrix}.$$

Since

$$\begin{vmatrix} \mathbf{0}_{n\times n} & I_n & \mathbf{0}_{n\times 1} \\ I_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & 1 \end{vmatrix} = -1,$$

this gives, applying Lemma (3 by 3), that

$$|\overline{H_{2n}^{2p+1}(\mathbf{g}^1)}| = -2|\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)| \cdot |\overline{\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)}| - 2|\overline{\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)}|^2 \equiv 0.$$

For (vii') and (vii") we will use the well–known (see [APWW, Remark 2.1] or [Bre, Page 96]) recurrence

$$|H_n^p(\mathbf{u})| \cdot |H_n^{p+2}(\mathbf{u})| - |H_n^{p+1}(\mathbf{u})|^2 = |H_{n-1}^{p+2}(\mathbf{u})| \cdot |H_{n+1}^p(\mathbf{u})|, \tag{9}$$

with $p \mapsto 2p$, $n \mapsto 2n+1$ and $\mathbf{u} = \mathbf{g}^1$, to get

$$|H_{2n+1}^{2p}(\mathbf{g}^1)| \cdot |H_{2n+1}^{2(p+1)}(\mathbf{g}^1)| - |H_{2n+1}^{2p+1}(\mathbf{g}^1)|^2 = |H_{2n}^{2(p+1)}(\mathbf{g}^1)| \cdot |H_{2(n+1)}^{2p}(\mathbf{g}^1)|.$$

Solving for $|H_{2n+1}^{2p+1}(\mathbf{g}^1)|$ and remembering that we are always taking everything modulo 2, this gives

$$|H_{2n+1}^{2p+1}(\mathbf{g}^1)| \equiv |H_{2n+1}^{2p}(\mathbf{g}^1)| \cdot |H_{2n+1}^{2(p+1)}(\mathbf{g}^1)| - |H_{2n}^{2(p+1)}(\mathbf{g}^1)| \cdot |H_{2(n+1)}^{2p}(\mathbf{g}^1)|.$$

We now must differentiate between the cases $p = 0$ and $p \geqslant 1$.

When $p = 0$, substituting in the identities of parts (iii”) and (ii”), we have

$$\begin{aligned}
|H_{2n+1}^{1}(\mathbf{g}^1)| \equiv \Big[&\Big(|H_{n+1}^{0}(\mathbf{g}^0)| \cdot |H_{n}^{1}(\mathbf{g}^1)| - |\overline{H_{n+1}^{0}(\mathbf{g}^0)}| \cdot |H_{n}^{1}(\mathbf{g}^1)| \\
&\qquad - |H_{n+1}^{0}(\mathbf{g}^0)| \cdot |\overline{H_{n}^{1}(\mathbf{g}^1)}|\Big) \\
&\times \Big(|H_{n+1}^{1}(\mathbf{g}^1)| \cdot |H_{n}^{2}(\mathbf{g}^1)| - |\overline{H_{n+1}^{1}(\mathbf{g}^1)}| \cdot |H_{n}^{2}(\mathbf{g}^1)| \\
&\qquad - |H_{n+1}^{1}(\mathbf{g}^1)| \cdot |\overline{H_{n}^{2}(\mathbf{g}^1)}|\Big)\Big] \\
- \Big[&\Big(|H_{n}^{1}(\mathbf{g}^1)| \cdot |H_{n}^{2}(\mathbf{g}^1)| - |\overline{H_{n}^{1}(\mathbf{g}^1)}| \cdot |H_{n}^{2}(\mathbf{g}^1)| - |H_{n}^{1}(\mathbf{g}^1)| \cdot |\overline{H_{n}^{2}(\mathbf{g}^1)}|\Big) \\
&\times \Big(|H_{n+1}^{0}(\mathbf{g}^0)| \cdot |H_{n+1}^{1}(\mathbf{g}^1)| - |\overline{H_{n+1}^{0}(\mathbf{g}^0)}| \cdot |H_{n+1}^{1}(\mathbf{g}^1)| \\
&\qquad - |H_{n+1}^{0}(\mathbf{g}^0)| \cdot |\overline{H_{n+1}^{1}(\mathbf{g}^1)}|\Big)\Big].
\end{aligned}$$

The similar result holds for $|H_{2n+1}^{1}(\mathbf{g}^0)|$ by replacing $\mathbf{g}^1$ with $\mathbf{g}^0$ in the above argument. This proves (vii’).

For $p \geqslant 1$, substituting in the identities of parts (iii”) and (ii”) and simplifying, this becomes

$$\begin{aligned}
|H_{2n+1}^{2p+1}(\mathbf{g}^1)| \equiv \Big[&\Big(|H_{n+1}^{p}(\mathbf{g}^1)| \cdot |H_{n}^{p+1}(\mathbf{g}^1)| - |\overline{H_{n+1}^{p}(\mathbf{g}^1)}| \cdot |H_{n}^{p+1}(\mathbf{g}^1)| \\
&\qquad - |H_{n+1}^{p}(\mathbf{g}^1)| \cdot |\overline{H_{n}^{p+1}(\mathbf{g}^1)}|\Big) \\
&\times \Big(|H_{n+1}^{p+1}(\mathbf{g}^1)| \cdot |H_{n}^{p+2}(\mathbf{g}^1)| - |\overline{H_{n+1}^{p+1}(\mathbf{g}^1)}| \cdot |H_{n}^{p+2}(\mathbf{g}^1)| \\
&\qquad - |H_{n+1}^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_{n}^{p+2}(\mathbf{g}^1)}|\Big)\Big] \\
- \Big[&\Big(|H_{n}^{p+1}(\mathbf{g}^1)| \cdot |H_{n}^{p+2}(\mathbf{g}^1)| - |\overline{H_{n}^{p+1}(\mathbf{g}^1)}| \cdot |H_{n}^{p+2}(\mathbf{g}^1)| \\
&\qquad - |H_{n}^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_{n}^{p+2}(\mathbf{g}^1)}|\Big) \\
&\times \Big(|H_{n+1}^{p}(\mathbf{g}^1)| \cdot |H_{n+1}^{p+1}(\mathbf{g}^1)| - |\overline{H_{n+1}^{p}(\mathbf{g}^1)}| \cdot |H_{n+1}^{p+1}(\mathbf{g}^1)| \\
&\qquad - |H_{n+1}^{p}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{p+1}(\mathbf{g}^1)}|\Big)\Big] \\
= \Big(&|\overline{H_{n+1}^{p}(\mathbf{g}^1)}| \cdot |H_{n}^{p+2}(\mathbf{g}^1)| - |H_{n+1}^{p}(\mathbf{g}^1)| \cdot |\overline{H_{n}^{p+2}(\mathbf{g}^1)}|\Big) \\
&\quad \times \Big(|H_{n}^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{p+1}(\mathbf{g}^1)}| - |\overline{H_{n}^{p+1}(\mathbf{g}^1)}| \cdot |H_{n+1}^{p+1}(\mathbf{g}^1)|\Big).
\end{aligned}$$

This proves (vii”).

For (viii’) and (viii”), we note that for all n, p and $\mathbf{u}$ we have, denoting $\mathbf{u}+1 = \{u(n)+1\}_{n\geqslant 0}$, that

$$H_n^p(\mathbf{u}+1) = \mathbf{1}_{n\times n} + H_n^p(\mathbf{u}).$$

Applying (9) for the sequence $\mathbf{g}^1+1$ and doing everything modulo 2, we have that

$$\begin{aligned}|\mathbf{1}_{n\times n} + H_n^{p+1}(\mathbf{g}^1)| &\equiv |\mathbf{1}_{n\times n} + H_n^{p}(\mathbf{g}^1)| \cdot |\mathbf{1}_{n\times n} + H_n^{p+2}(\mathbf{g}^1)| \\ &\quad + |\mathbf{1}_{(n-1)\times(n-1)} + H_{n-1}^{p+2}(\mathbf{g}^1)| \cdot |\mathbf{1}_{(n+1)\times(n+1)} + H_{n+1}^{p}(\mathbf{g}^1)|.\end{aligned}$$

Now sending $p \mapsto 2p$ and $n \mapsto 2n+1$ yields

$$\begin{aligned}&|\mathbf{1}_{(2n+1)\times(2n+1)} + H_{2n+1}^{2p+1}(\mathbf{g}^1)| \\ &\quad\equiv |\mathbf{1}_{(2n+1)\times(2n+1)} + H_{2n+1}^{2p}(\mathbf{g}^1)| \cdot |\mathbf{1}_{(2n+1)\times(2n+1)} + H_{2n+1}^{2(p+1)}(\mathbf{g}^1)| \\ &\qquad + |\mathbf{1}_{2n\times 2n} + H_{2n}^{2(p+1)}(\mathbf{g}^1)| \cdot |\mathbf{1}_{2(n+2)\times 2(n+2)} + H_{2(n+2)}^{2p}(\mathbf{g}^1)|.\end{aligned}$$

Applying the identity from Lemma 6(i) and solving for $|\overline{H_{2n+1}^{2p+1}(\mathbf{g}^1)}|$, gives

$$\begin{aligned}|\overline{H_{2n+1}^{2p+1}(\mathbf{g}^1)}| &\equiv |H_{2n+1}^{2p+1}(\mathbf{g}^1)| \\ &\quad + \left(|H_{2n+1}^{2p}(\mathbf{g}^1)| + |\overline{H_{2n+1}^{2p}(\mathbf{g}^1)}|\right) \cdot \left(|H_{2n+1}^{2(p+1)}(\mathbf{g}^1)| + |\overline{H_{2n+1}^{2(p+1)}(\mathbf{g}^1)}|\right) \\ &\quad + \left(|H_{2n}^{2(p+1)}(\mathbf{g}^1)| + |\overline{H_{2n}^{2(p+1)}(\mathbf{g}^1)}|\right) \cdot \left(|H_{2(n+2)}^{2p}(\mathbf{g}^1)| + |\overline{H_{2(n+2)}^{2p}(\mathbf{g}^1)}|\right).\end{aligned}$$

Now applying the results we have just proven from (i”), (ii”), (iii”), (iv”), and (vii”) we have that

$$\begin{aligned}|\overline{H_{2n+1}^{2p+1}(\mathbf{g}^1)}| &\equiv \left(|H_n^{p+2}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{p}(\mathbf{g}^1)}| - |H_{n+1}^{p}(\mathbf{g}^1)| \cdot |\overline{H_n^{p+2}(\mathbf{g}^1)}|\right) \\ &\quad \times \left(|H_n^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{p+1}(\mathbf{g}^1)}| - |H_{n+1}^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_n^{p+1}(\mathbf{g}^1)}|\right) \\ &\quad + \left(|H_{2n+1}^{2p}(\mathbf{g}^1)| + |\overline{H_{2n+1}^{2p}(\mathbf{g}^1)}|\right) \cdot \left(|H_{n+1}^{p+1}(\mathbf{g}^1)| \cdot |H_n^{p+2}(\mathbf{g}^1)|\right) \\ &\quad + \left(|H_n^{p+1}(\mathbf{g}^1)| \cdot |H_n^{p+2}(\mathbf{g}^1)|\right) \cdot \left(|H_{2(n+2)}^{2p}(\mathbf{g}^1)| + |\overline{H_{2(n+2)}^{2p}(\mathbf{g}^1)}|\right).\end{aligned} \tag{10}$$

We note differentiate between $p=0$ and $p \geqslant 1$.

If $p=0$, we apply (i’), (ii’), (iii’), and (iv’) to equivalence (10) to get

$$\begin{aligned}|\overline{H_{2n+1}^{1}(\mathbf{g}^1)}| &\equiv \left(|H_n^{2}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{0}(\mathbf{g}^1)}| - |H_{n+1}^{0}(\mathbf{g}^1)| \cdot |\overline{H_n^{2}(\mathbf{g}^1)}|\right) \\ &\quad \times \left(|H_n^{1}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{1}(\mathbf{g}^1)}| - |H_{n+1}^{1}(\mathbf{g}^1)| \cdot |\overline{H_n^{1}(\mathbf{g}^1)}|\right) \\ &\quad + |H_{n+1}^{0}(\mathbf{g}^0)| \cdot |H_n^{1}(\mathbf{g}^1)| \cdot |H_{n+1}^{1}(\mathbf{g}^1)| \cdot |H_n^{2}(\mathbf{g}^1)| \\ &\quad + |H_n^{1}(\mathbf{g}^1)| \cdot |H_n^{2}(\mathbf{g}^1)| \cdot |H_{n+2}^{0}(\mathbf{g}^0)| \cdot |\overline{H_{n+2}^{1}(\mathbf{g}^1)}|,\end{aligned}$$

which proves (viii’).

If $p \geqslant 1$, we apply (i"), (ii"), (iii"), and (iv") to equivalence (10) to get

$$\begin{aligned}
|\overline{H_{2n+1}^{2p+1}(\mathbf{g}^1)}| \equiv &\left(|H_n^{p+2}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{p}(\mathbf{g}^1)}| - |H_{n+1}^{p}(\mathbf{g}^1)| \cdot |\overline{H_n^{p+2}(\mathbf{g}^1)}|\right)\\
&\times \left(|H_n^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_{n+1}^{p+1}(\mathbf{g}^1)}| - |H_{n+1}^{p+1}(\mathbf{g}^1)| \cdot |\overline{H_n^{p+1}(\mathbf{g}^1)}|\right)\\
&+ |H_{n+1}^{p}(\mathbf{g}^1)| \cdot |H_n^{p+1}(\mathbf{g}^1)| \cdot |H_{n+1}^{p+1}(\mathbf{g}^1)| \cdot |H_n^{p+2}(\mathbf{g}^1)|\\
&+ |H_n^{p+1}(\mathbf{g}^1)| \cdot |H_n^{p+2}(\mathbf{g}^1)| \cdot |H_{n+2}^{p}(\mathbf{g}^1)| \cdot |H_{n+2}^{p+1}(\mathbf{g}^1)|,
\end{aligned}$$

which proves (viii") and completes the proof of the lemma. □

We have the following corollary. Note that we have made enumerated the parts of the corollary to coincide with the enumeration of Lemma 7.

Corollary 8. *For all $n \geqslant 1$, we have*

(i') $|H_{2n}^0(\mathbf{g}^1)| \equiv |H_n^0(\mathbf{g}^0)| \cdot |H_n^1(\mathbf{g}^1)| - |\overline{H_n^0(\mathbf{g}^0)}| \cdot |H_n^1(\mathbf{g}^1)| - |H_n^0(\mathbf{g}^0)| \cdot |\overline{H_n^1(\mathbf{g}^1)}|$,
$|H_{2n}^0(\mathbf{g}^0)| \equiv |H_n^0(\mathbf{g}^1)| \cdot |H_n^1(\mathbf{g}^1)| - |\overline{H_n^0(\mathbf{g}^1)}| \cdot |H_n^1(\mathbf{g}^1)| - |H_n^0(\mathbf{g}^1)| \cdot |\overline{H_n^1(\mathbf{g}^1)}|$,

(i") $|H_{2n}^2(\mathbf{g}^1)| \equiv |H_n^1(\mathbf{g}^1)| \cdot |H_n^2(\mathbf{g}^1)| - |\overline{H_n^1(\mathbf{g}^1)}| \cdot |H_n^2(\mathbf{g}^1)| - |H_n^1(\mathbf{g}^1)| \cdot |\overline{H_n^2(\mathbf{g}^1)}|$,

(ii') $|\overline{H_{2n}^0(\mathbf{g}^1)}| \equiv |H_n^0(\mathbf{g}^0)| \cdot |\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_n^0(\mathbf{g}^0)}| \cdot |H_n^1(\mathbf{g}^1)|$,
$|\overline{H_{2n}^0(\mathbf{g}^0)}| \equiv |H_n^0(\mathbf{g}^1)| \cdot |\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_n^0(\mathbf{g}^1)}| \cdot |H_n^1(\mathbf{g}^1)|$,

(ii") $|\overline{H_{2n}^2(\mathbf{g}^1)}| \equiv |H_n^1(\mathbf{g}^1)| \cdot |\overline{H_n^2(\mathbf{g}^1)}| + |\overline{H_n^1(\mathbf{g}^1)}| \cdot |H_n^2(\mathbf{g}^1)|$,

(iii') $|H_{2n+1}^0(\mathbf{g}^1)| \equiv |H_{n+1}^0(\mathbf{g}^0)| \cdot |H_n^1(\mathbf{g}^1)| - |\overline{H_{n+1}^0(\mathbf{g}^0)}| \cdot |H_n^1(\mathbf{g}^1)|$
$\quad - |H_{n+1}^0(\mathbf{g}^0)| \cdot |\overline{H_n^1(\mathbf{g}^1)}|$
$|H_{2n+1}^0(\mathbf{g}^0)| = |H_{n+1}^0(\mathbf{g}^1)| \cdot |H_n^1(\mathbf{g}^1)| - |\overline{H_{n+1}^0(\mathbf{g}^1)}| \cdot |H_n^1(\mathbf{g}^1)|$
$\quad - |H_{n+1}^0(\mathbf{g}^1)| \cdot |\overline{H_n^1(\mathbf{g}^1)}|$

(iii") $|H_{2n+1}^2(\mathbf{g}^1)| \equiv |H_{n+1}^1(\mathbf{g}^1)| \cdot |H_n^2(\mathbf{g}^1)| - |\overline{H_{n+1}^1(\mathbf{g}^1)}| \cdot |H_n^2(\mathbf{g}^1)|$
$\quad - |H_{n+1}^1(\mathbf{g}^1)| \cdot |\overline{H_n^2(\mathbf{g}^1)}|$,

(iv') $|\overline{H_{2n+1}^0(\mathbf{g}^1)}| \equiv |H_{n+1}^0(\mathbf{g}^0)| \cdot |\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_{n+1}^0(\mathbf{g}^0)}| \cdot |H_n^1(\mathbf{g}^1)|$
$|\overline{H_{2n+1}^0(\mathbf{g}^0)}| \equiv |H_{n+1}^0(\mathbf{g}^1)| \cdot |\overline{H_n^1(\mathbf{g}^1)}| + |\overline{H_{n+1}^0(\mathbf{g}^1)}| \cdot |H_n^1(\mathbf{g}^1)|$,

(iv") $|\overline{H_{2n+1}^2(\mathbf{g}^1)}| \equiv |H_{n+1}^1(\mathbf{g}^1)| \cdot |\overline{H_n^2(\mathbf{g}^1)}| + |\overline{H_{n+1}^1(\mathbf{g}^1)}| \cdot |H_n^2(\mathbf{g}^1)|$,

(v) $|H_{2n}^1(\mathbf{g}^1)| \equiv |H_n^1(\mathbf{g}^1)|$,

(vi) $|\overline{H_{2n}^1(\mathbf{g}^1)}| \equiv 0$,

$$\begin{aligned}
\text{(vii')}\ \ |H^1_{2n+1}(\mathbf{g}^1)| \equiv \Big[& \Big(|H^0_{n+1}(\mathbf{g}^0)| \cdot |H^1_n(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^0)}| \cdot |H^1_n(\mathbf{g}^1)| \\
& \quad - |H^0_{n+1}(\mathbf{g}^0)| \cdot |\overline{H^1_n(\mathbf{g}^1)}| \Big) \\
& \times \Big(|H^1_{n+1}(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_{n+1}(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| \\
& \quad - |H^1_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}| \Big) \Big] \\
& - \Big[\Big(|H^1_n(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_n(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| \\
& \quad - |H^1_n(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}| \Big) \\
& \times \Big(|H^0_{n+1}(\mathbf{g}^0)| \cdot |H^1_{n+1}(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^0)}| \cdot |H^1_{n+1}(\mathbf{g}^1)| \\
& \quad - |H^0_{n+1}(\mathbf{g}^0)| \cdot |\overline{H^1_{n+1}(\mathbf{g}^1)}| \Big) \Big],
\end{aligned}$$

$$\begin{aligned}
|H^1_{2n+1}(\mathbf{g}^0)| \equiv \Big[& \Big(|H^0_{n+1}(\mathbf{g}^1)| \cdot |H^1_n(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^1)}| \cdot |H^1_n(\mathbf{g}^1)| \\
& \quad - |H^0_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^1_n(\mathbf{g}^1)}| \Big) \\
& \times \Big(|H^1_{n+1}(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_{n+1}(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| \\
& \quad - |H^1_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}| \Big) \Big] \\
& - \Big[\Big(|H^1_n(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| - |\overline{H^1_n(\mathbf{g}^1)}| \cdot |H^2_n(\mathbf{g}^1)| \\
& \quad - |H^1_n(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}| \Big) \\
& \times \Big(|H^0_{n+1}(\mathbf{g}^1)| \cdot |H^1_{n+1}(\mathbf{g}^1)| - |\overline{H^0_{n+1}(\mathbf{g}^1)}| \cdot |H^1_{n+1}(\mathbf{g}^1)| \\
& \quad - |H^0_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^1_{n+1}(\mathbf{g}^1)}| \Big) \Big],
\end{aligned}$$

$$\begin{aligned}
\text{(viii')}\ \ |\overline{H^1_{2n+1}(\mathbf{g}^1)}| \equiv & \Big(|H^2_n(\mathbf{g}^1)| \cdot |\overline{H^0_{n+1}(\mathbf{g}^1)}| - |H^0_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^2_n(\mathbf{g}^1)}| \Big) \\
& \times \Big(|H^1_n(\mathbf{g}^1)| \cdot |\overline{H^1_{n+1}(\mathbf{g}^1)}| - |H^1_{n+1}(\mathbf{g}^1)| \cdot |\overline{H^1_n(\mathbf{g}^1)}| \Big) \\
& + |H^0_{n+1}(\mathbf{g}^0)| \cdot |H^1_n(\mathbf{g}^1)| \cdot |H^1_{n+1}(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| \\
& + |H^1_n(\mathbf{g}^1)| \cdot |H^2_n(\mathbf{g}^1)| \cdot |H^0_{n+2}(\mathbf{g}^0)| \cdot |\overline{H^1_{n+2}(\mathbf{g}^1)}|.
\end{aligned}$$

Proof of Theorem 3. First let us note that for $p \geqslant 1$, we trivially have that $H^p_n(\mathbf{g}^0) = H^p_n(\mathbf{g}^1)$, so that we do not need to worry about proving separately the cases for $\mathbf{g}^1$ and $\mathbf{g}^0$ in this range.

We easily check (say with MAPLE) that Theorem 3 is true for $n \leqslant 12$. The rest of the proof now follows by breaking up the cases of n modulo 6; that is, check that the theorem is true for n equal to $6k, 6k+1, 6k+2, 6k+3, 6k+4$, and $6k+5$. We write here only the case when $n = 6k$. All of the other cases follow *mutatis mutandis*.

To this end, suppose the theorem is true for all $6k < m$. If $m = 6k$ for some k then Corollary 8(i') gives

$$\begin{aligned}|H^0_{12k}(\mathbf{g}^1)| &\equiv |H^0_{6k}(\mathbf{g}^0)| \cdot |H^1_{6k}(\mathbf{g}^1)| - |\overline{H^0_{6k}(\mathbf{g}^0)}| \cdot |H^1_{6k}(\mathbf{g}^1)| - |H^0_{6k}(\mathbf{g}^0)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}| \\ &\equiv 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 0 \equiv 0,\end{aligned}$$

and

$$\begin{aligned}|H^0_{12k}(\mathbf{g}^0)| &= |H^0_{6k}(\mathbf{g}^1)| \cdot |H^1_{6k}(\mathbf{g}^1)| - |\overline{H^0_{6k}(\mathbf{g}^1)}| \cdot |H^1_{6k}(\mathbf{g}^1)| - |H^0_{6k}(\mathbf{g}^1)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}| \\ &\equiv 0 \cdot 1 - 1 \cdot 1 - 0 \cdot 0 \equiv 1.\end{aligned}$$

Corollary 8(i") gives

$$\begin{aligned}|H^2_{12k}(\mathbf{g}^1)| &\equiv |H^1_{6k}(\mathbf{g}^1)| \cdot |H^2_{6k}(\mathbf{g}^1)| - |\overline{H^1_{6k}(\mathbf{g}^1)}| \cdot |H^2_{6k}(\mathbf{g}^1)| - |H^1_{6k}(\mathbf{g}^1)| \cdot |\overline{H^2_{6k}(\mathbf{g}^1)}| \\ &\equiv 1 \cdot 1 - 0 \cdot 1 - 1 \cdot 0 \equiv 1.\end{aligned}$$

Corollary 8(ii') gives

$$\begin{aligned}|\overline{H^0_{12k}(\mathbf{g}^1)}| &\equiv |H^0_{6k}(\mathbf{g}^0)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}| + |\overline{H^0_{6k}(\mathbf{g}^0)}| \cdot |H^1_{6k}(\mathbf{g}^1)| \\ &\equiv 1 \cdot 0 + 1 \cdot 1 \equiv 1,\end{aligned}$$

and

$$\begin{aligned}|\overline{H^0_{12k}(\mathbf{g}^0)}| &\equiv |H^0_{6k}(\mathbf{g}^1)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}| + |\overline{H^0_{6k}(\mathbf{g}^1)}| \cdot |H^1_{6k}(\mathbf{g}^1)| \\ &\equiv 0 \cdot 0 + 1 \cdot 1 \equiv 1.\end{aligned}$$

Corollary 8(ii") gives

$$\begin{aligned}|\overline{H^2_{12k}(\mathbf{g}^1)}| &\equiv |H^1_{6k}(\mathbf{g}^1)| \cdot |\overline{H^2_{6k}(\mathbf{g}^1)}| + |\overline{H^1_{6k}(\mathbf{g}^1)}| \cdot |H^2_{6k}(\mathbf{g}^1)| \\ &\equiv 1 \cdot 0 + 0 \cdot 1 \equiv 0.\end{aligned}$$

Corollary 8(iii') gives

$$\begin{aligned}|H^0_{12k+1}(\mathbf{g}^1)| &\equiv |H^0_{6k+1}(\mathbf{g}^0)| \cdot |H^1_{6k}(\mathbf{g}^1)| - |\overline{H^0_{6k+1}(\mathbf{g}^0)}| \cdot |H^1_{6k}(\mathbf{g}^1)| \\ &\qquad - |H^0_{6k+1}(\mathbf{g}^0)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}| \\ &\equiv 0 \cdot 1 - 1 \cdot 1 - 0 \cdot 0 \equiv 1,\end{aligned}$$

and

$$\begin{aligned}|H^0_{12k+1}(\mathbf{g}^0)| &\equiv |H^0_{6k+1}(\mathbf{g}^1)| \cdot |H^1_{6k}(\mathbf{g}^1)| - |\overline{H^0_{6k+1}(\mathbf{g}^1)}| \cdot |H^1_{6k}(\mathbf{g}^1)| \\ &\qquad - |H^0_{6k+1}(\mathbf{g}^1)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}| \\ &\equiv 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 0 \equiv 0.\end{aligned}$$

Corollary 8(iii") gives

$$\begin{aligned}|H^2_{12k+1}(\mathbf{g}^1)| &\equiv |H^1_{6k+1}(\mathbf{g}^1)| \cdot |H^2_{6k}(\mathbf{g}^1)| - |\overline{H^1_{6k+1}(\mathbf{g}^1)}| \cdot |H^2_{6k}(\mathbf{g}^1)| \\ &\qquad - |H^1_{6k+1}(\mathbf{g}^1)| \cdot |\overline{H^2_{6k}(\mathbf{g}^1)}| \\ &\equiv 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 0 \equiv 0.\end{aligned}$$

Corollary 8(iv') gives

$$\begin{aligned}|\overline{H^0_{12k+1}(\mathbf{g}^1)}| &\equiv |H^0_{6k+1}(\mathbf{g}^0)|\cdot|\overline{H^1_{6k}(\mathbf{g}^1)}| + |\overline{H^0_{6k+1}(\mathbf{g}^0)}|\cdot|H^1_{6k}(\mathbf{g}^1)|\\ &\equiv 0\cdot 0 + 1\cdot 1 \equiv 1.\end{aligned}$$

and

$$\begin{aligned}|\overline{H^0_{12k+1}(\mathbf{g}^0)}| &\equiv |H^0_{6k+1}(\mathbf{g}^1)|\cdot|\overline{H^1_{6k}(\mathbf{g}^1)}| + |\overline{H^0_{6k+1}(\mathbf{g}^1)}|\cdot|H^1_{6k}(\mathbf{g}^1)|\\ &\equiv 1\cdot 0 + 1\cdot 1 \equiv 1.\end{aligned}$$

Corollary 8(iv") gives

$$\begin{aligned}|\overline{H^2_{12k+1}(\mathbf{g}^1)}| &\equiv |H^1_{6k+1}(\mathbf{g}^1)|\cdot|\overline{H^2_{6k}(\mathbf{g}^1)}| + |\overline{H^1_{6k+1}(\mathbf{g}^1)}|\cdot|H^2_{6k}(\mathbf{g}^1)|\\ &\equiv 1\cdot 0 + 1\cdot 1 \equiv 1.\end{aligned}$$

Corollary 8(v) gives $|H^1_{12k}(\mathbf{g}^1)| \equiv |H^1_{6k}(\mathbf{g}^1)| \equiv 1$.

Corollary 8(vi) gives $|\overline{H^1_{12k}(\mathbf{g}^1)}| \equiv 0$.
Corollary 8(vii') gives

$$\begin{aligned}|H^1_{12k+1}(\mathbf{g}^1)| \equiv \Big[&\Big(|H^0_{6k+1}(\mathbf{g}^0)|\cdot|H^1_{6k}(\mathbf{g}^1)| - |\overline{H^0_{6k+1}(\mathbf{g}^0)}|\cdot|H^1_{6k}(\mathbf{g}^1)|\\ &\qquad - |H^0_{6k+1}(\mathbf{g}^0)|\cdot|\overline{H^1_{6k}(\mathbf{g}^1)}|\Big)\\ &\times\Big(|H^1_{6k+1}(\mathbf{g}^1)|\cdot|H^2_{6k}(\mathbf{g}^1)| - |\overline{H^1_{6k+1}(\mathbf{g}^1)}|\cdot|H^2_{6k}(\mathbf{g}^1)|\\ &\qquad - |H^1_{6k+1}(\mathbf{g}^1)|\cdot|\overline{H^2_{6k}(\mathbf{g}^1)}|\Big)\Big]\\ &- \Big[\Big(|H^1_{6k}(\mathbf{g}^1)|\cdot|H^2_{6k}(\mathbf{g}^1)| - |\overline{H^1_{6k}(\mathbf{g}^1)}|\cdot|H^2_{6k}(\mathbf{g}^1)| - |H^1_{6k}(\mathbf{g}^1)|\cdot|\overline{H^2_{6k}(\mathbf{g}^1)}|\Big)\\ &\times\Big(|H^0_{6k+1}(\mathbf{g}^0)|\cdot|H^1_{6k+1}(\mathbf{g}^1)| - |\overline{H^0_{6k+1}(\mathbf{g}^0)}|\cdot|H^1_{6k+1}(\mathbf{g}^1)|\\ &\qquad - |H^0_{6k+1}(\mathbf{g}^0)|\cdot|\overline{H^1_{6k}(\mathbf{g}^1)}|\Big)\Big]\\ &\equiv (0\cdot 1 - 1\cdot 1 - 0\cdot 0)(1\cdot 1 - 1\cdot 1 - 1\cdot 0)\\ &\qquad - (1\cdot 1 - 0\cdot 1 - 1\cdot 0)(0\cdot 1 - 1\cdot 1 - 0\cdot 1)\\ &\equiv 1,\end{aligned}$$

and

$$\begin{aligned}
|H^1_{12k+1}(\mathbf{g}^0)| \equiv \Big[&\Big(|H^0_{6k+1}(\mathbf{g}^1)| \cdot |H^1_{6k}(\mathbf{g}^1)| - |\overline{H^0_{6k+1}(\mathbf{g}^1)}| \cdot |H^1_{6k}(\mathbf{g}^1)| \\
&\qquad - |H^0_{6k+1}(\mathbf{g}^1)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}|\Big) \\
&\times \Big(|H^1_{6k+1}(\mathbf{g}^1)| \cdot |H^2_{6k}(\mathbf{g}^1)| - |\overline{H^1_{6k+1}(\mathbf{g}^1)}| \cdot |H^2_{6k}(\mathbf{g}^1)| \\
&\qquad - |H^1_{6k+1}(\mathbf{g}^1)| \cdot |\overline{H^2_{6k}(\mathbf{g}^1)}|\Big)\Big] \\
&- \Big[\Big(|H^1_{6k}(\mathbf{g}^1)| \cdot |H^2_{6k}(\mathbf{g}^1)| - |\overline{H^1_{6k}(\mathbf{g}^1)}| \cdot |H^2_{6k}(\mathbf{g}^1)| \\
&\qquad - |H^1_{6k}(\mathbf{g}^1)| \cdot |\overline{H^2_{6k}(\mathbf{g}^1)}|\Big) \\
&\times \Big(|H^0_{6k+1}(\mathbf{g}^1)| \cdot |H^1_{6k+1}(\mathbf{g}^1)| - |\overline{H^0_{6k+1}(\mathbf{g}^1)}| \cdot |H^1_{6k+1}(\mathbf{g}^1)| \\
&\qquad - |H^0_{6k+1}(\mathbf{g}^1)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}|\Big)\Big] \\
&\equiv (1\cdot 1 - 1\cdot 1 - 1\cdot 0)(1\cdot 1 - 1\cdot 1 - 1\cdot 0) \\
&\qquad - (1\cdot 1 - 0\cdot 1 - 1\cdot 0)(1\cdot 1 - 1\cdot 1 - 1\cdot 1) \\
&\equiv 1.
\end{aligned}$$

Corollary 8(viii') gives

$$\begin{aligned}
|\overline{H^1_{12k+1}(\mathbf{g}^1)}| \equiv &\Big(|H^2_{6k}(\mathbf{g}^1)| \cdot |\overline{H^0_{6k+1}(\mathbf{g}^1)}| - |H^0_{6k+1}(\mathbf{g}^1)| \cdot |\overline{H^2_{6k}(\mathbf{g}^1)}|\Big) \\
&\quad \times \Big(|H^1_{6k}(\mathbf{g}^1)| \cdot |\overline{H^1_{6k+1}(\mathbf{g}^1)}| - |H^1_{6k+1}(\mathbf{g}^1)| \cdot |\overline{H^1_{6k}(\mathbf{g}^1)}|\Big) \\
&\quad + |H^0_{6k+1}(\mathbf{g}^0)| \cdot |H^1_{6k}(\mathbf{g}^1)| \cdot |H^1_{6k+1}(\mathbf{g}^1)| \cdot |H^2_{6k}(\mathbf{g}^1)| \\
&\quad + |H^1_{6k}(\mathbf{g}^1)| \cdot |H^2_{6k}(\mathbf{g}^1)| \cdot |H^0_{6k+2}(\mathbf{g}^0)| \cdot |\overline{H^1_{6k+2}(\mathbf{g}^1)}| \\
\equiv &(1\cdot 1 - 1\cdot 0)(1\cdot 1 - 1\cdot 0) + 0\cdot 1\cdot 1\cdot 0 + 1\cdot 0\cdot 1\cdot 0 \\
\equiv &1.
\end{aligned}$$

□

We can easily relate Theorem 3 to give a similar result for the sequence $\mathbf{f}$ of coefficients of the series $\mathcal{F}(z) = \sum_{n\geqslant 0} z^{2^n}(1+z^{2^n})^{-1}$.

Corollary 9. *Let $\mathbf{h} = \{h(n)\}_{n\geqslant 1}$ be a sequence which is equivalent modulo 2 to $\mathbf{g}$; that is, $h(n) \equiv g(n) \pmod 2$. Then $|H^1_n(\mathbf{h})|$ is nonzero for all $n \geqslant 1$. In particular, the determinant of $|H^1_n(\mathbf{f})|$ is nonzero for all $n \geqslant 1$.*

Proof. It is enough to note that since $h(n) \equiv g(n) \pmod 2$ for all $n \geqslant 1$, and so modulo 2 we have $H^1_n(\mathbf{h}) \equiv H^1_n(\mathbf{g}^1)$. □

Proof of Theorem 2. This is a direct consequence of Theorem 3 and Corollary 9. □

3. Rational approximation of values of Golomb's series

Given an analytic function $F(z)$, the rational function $R(x)$, with the degree of the numerator bounded by m and the degree of the denominator bounded by n, is the $[m/n]_F$ *Padé approximant* to $F(z)$ provided

$$F(z) - R(z) = O(z^{m+n+1}).$$

We will need the following lemma connecting Hankel determinants to Padé approximants (see [Bre, Page 35]).

Lemma 10 (Brezinski [Bre])**.** *Let* $\mathbf{c} = \{c(n)\}_{n\geqslant 0}$ *and* $\mathcal{C}(z) = \sum_{n\geqslant 0} c(n)z^n \in \mathbb{Z}[[z]]$*. If* $\det H_k^0(\mathbf{c}) \neq 0$ *for all* $k \geqslant 1$*, then the Padé approximant* $[k-1/k]_{\mathcal{C}}$ *exists and satisfies*

$$\mathcal{C}(z) - [k-1/k]_{\mathcal{C}} = \frac{\det H_{k+1}^0(\mathbf{c})}{\det H_k^0(\mathbf{c})} z^{2k} + O(z^{2k+1}).$$

An immediate consequence of Theorem 3 and the Lemma 10 is the following lemma.

Lemma 11. *Let* $k \geqslant 1$*,* $\mathbf{h} = \{h(n)\}_{n\geqslant 1}$ *be a sequence for which* $h(n) \equiv g(n) \pmod 2$*, and* $\mathcal{H}(z) := \sum_{n\geqslant 1} h(n)z^n$*. Then there exists a nonzero rational number* h_k *and polynomials* $P_k(z), Q_k(z) \in \mathbb{Z}[z]$ *with degrees bounded above by* k *such that*

$$\mathcal{H}(z) - \frac{P_k(z)}{Q_k(z)} = h_k z^{2k+1} + O(z^{2k+2}).$$

Proof. Define the function $\mathcal{H}'(z) := \mathcal{H}(z)/z = \sum_{n\geqslant 0} h'(n)z^n$. Then $h'(n) = h(n+1)$ for all $n \geqslant 0$, and by Corollary 9 for all $k \geqslant 1$ we have that

$$H_k^1(\mathbf{h}) = H_k^0(\mathbf{h}') \neq 0.$$

By Lemma 10 we have that the $[k-1/k]_{\mathcal{H}'}$ exists and satisfies

$$\mathcal{H}'(z) - [k-1/k]_{\mathcal{H}'} = \frac{\det H_{k+1}^1(\mathbf{h})}{\det H_k^1(\mathbf{h})} z^{2k} + O(z^{2k+1});$$

that is there exists polynomials $R_k(z), Q_k(z) \in \mathbb{Z}[z]$, with

$$\deg R_k(z) \leqslant k-1 \qquad \text{and} \qquad \deg Q_k(z) \leqslant k,$$

such that

$$\mathcal{H}'(z) - \frac{R_k(z)}{Q_k(z)} = h_k z^{2k} + O(z^{2k+1}), \tag{11}$$

where we have set $h_k := \det H_{k+1}^1(\mathbf{h})/\det H_k^1(\mathbf{h})$ which is nonzero, since each of the numerator and denominator are nonzero. Multiplying both sides of (11) by z and denoting $P_k(z) := zR_k(z)$ proves the lemma. □

Along with Lemma 11 we will need the following result of Adamczewski and Rivoal [AR, Lemma 4.1] and a modification of a lemma of Bugeaud [Bug, Lemma 2].

Lemma 12 (Adamczewski and Rivoal [AR]). *Let ξ, δ, ρ and ϑ be real numbers such that $0 < \delta \leqslant \rho$ and $\vartheta \geqslant 1$. Let us assume that there exists a sequence $\{p_n/q_n\}_{n\geqslant 1}$ of rational numbers and some positive constants c_0, c_1 and c_2 such that both*

$$q_n < q_{n+1} \leqslant c_0 q_n^{\vartheta},$$

and

$$\frac{c_1}{q_n^{1+\rho}} \leqslant \left|\xi - \frac{p_n}{q_n}\right| \leqslant \frac{c_2}{q_n^{1+\delta}}.$$

Then we have that

$$\mu(\xi) \leqslant (1+\rho)\frac{\vartheta}{\delta}.$$

Lemma 13 (Modified Bugeaud). *Let $K \geqslant 1$ and n_0 be positive integers. Let $(a_j)_{j\geqslant 1}$ be the increasing sequence of integers composed of all the numbers of the form $k2^n$, where $n \geqslant n_0$ and k ranges over all the odd integers in $[2^{K-1}+1, 2^K+1]$. Then*

$$a_{j+1} \leqslant \left(\frac{2^{K-1}+3}{2^{K-1}+1}\right) a_j.$$

Proof. Let n be large enough and consider the increasing sequence $(a_j)_{j\geqslant 1}$ of all integers of the form $k2^n$ where k is an odd number in $[2^{K-1}+1, 2^K+1]$. Note that for a given j we have that for some m and some odd number a with $1 \leqslant a \leqslant 2^{K-1}+1$ we have $a_j = 2^m(2^{K-1}+a)$. We consider two cases.

If $a < 2^{K-1}+1$, then $a_{j+1} \leqslant 2^m(2^{K-1}+a+2)$, so that

$$\frac{a_{j+1}}{a_j} \leq \frac{2^{K-1}+a+2}{2^{K-1}+a} \leqslant \frac{2^{K-1}+3}{2^{K-1}+1}.$$

If $a = 2^{K-1}+1$, then $a_{j+1} \leqslant 2^{m+1}(2^{K-1}+1) = 2^m(2^K+2)$, so that

$$\frac{a_{j+1}}{a_j} \leqslant \frac{2^K+2}{2^K+1} \leqslant \frac{2^{K-1}+3}{2^{K-1}+1}.$$

This proves the lemma. □

Proof of Theorem 1. Let $\epsilon \in \{-1, 1\}$ and set

$$\mathcal{H}(z) := \sum_{n\geqslant 0} \frac{z^{2^n}}{1+\epsilon z^{2^k}} = \sum_{n\geqslant 1} h(n) z^n.$$

Note here that $\mathcal{H}(z)$ satisfies the functional equation

$$\mathcal{H}(z^{2^m}) = \mathcal{H}(z) - \sum_{k=0}^{m-1} \frac{z^{2^k}}{1+\epsilon z^{2^k}}.$$

Applying Lemma 11, there exist polynomials $P_{k,0}(z), Q_{k,0}(z) \in \mathbb{Z}[z]$, with both $\deg P_{k,0}(z)$ and $\deg Q_{k,0}(z)$ at most k, and a nonzero $h_k \in \mathbb{Q}$ such that

$$\mathcal{H}(z) - \frac{P_{k,0}(z)}{Q_{k,0}(z)} = h_k z^{2k+1} + O(z^{2k+2}).$$

Thus sending $z \mapsto z^{2^m}$ we have that

$$\mathcal{H}(z^{2^m}) - \frac{P_{k,0}(z^{2^m})}{Q_{k,0}(z^{2^m})} = h_k z^{2^m(2k+1)} + O(z^{2^m(2k+2)}),$$

and so using the functional equation for $\mathcal{H}(z)$ we then have that

$$\mathcal{H}(z) - \left(\sum_{k=0}^{m-1} \frac{z^{2^k}}{1+\epsilon z^{2^k}} + \frac{P_{k,0}(z^{2^m})}{Q_{k,0}(z^{2^m})}\right) = h_k z^{2^m(2k+1)} + O(z^{2^m(2k+2)}).$$

Now define $P_{k,m}(z)$ and $Q_{k,m}(z)$ by

$$\frac{P_{k,m}(z)}{Q_{k,m}(z)} := \sum_{k=0}^{m-1} \frac{z^{2^k}}{1+\epsilon z^{2^k}} + \frac{P_{k,0}(z^{2^m})}{Q_{k,0}(z^{2^m})},$$

so that

$$\mathcal{H}(z) - \frac{P_{k,m}(z)}{Q_{k,m}(z)} = h_k z^{2^m(2k+1)} + O(z^{2^m(2k+2)}).$$

Now let $b \geqslant 2$ be an integer and as before set $z = \frac{1}{b}$. Then for $\varepsilon > 0$, we have for large enough m, say $m \geqslant m_0(k)$, that

$$(1-\varepsilon)h_k b^{-2^m(2k+1)} \leqslant \left|\mathcal{H}(1/b) - \frac{P_{k,m}(1/b)}{Q_{k,m}(1/b)}\right| \leqslant (1+\varepsilon)h_k b^{-2^m(2k+1)}.$$

To get the degrees of $P_{k,m}(z)$ and $Q_{k,m}(z)$ we write

$$\frac{P(z)}{Q(z)} = \sum_{k=0}^{m-1} \frac{z^{2^k}}{1+\epsilon z^{2^k}}. \tag{12}$$

Note that using (12) it is immediate that

$$\deg P(z) \leqslant \deg Q(z) \leqslant 2^m.$$

Using the definitions of $P(z)$ and $Q(z)$, we have that both

$$\deg Q_{k,m}(z) = \deg Q(z)Q_{k,0}(z^{2^m}) = \deg Q(z) + \deg Q_{k,0}(z^{2^m}) \leqslant 2^m(k+1),$$

and

$$\deg P_{k,m}(z) = \max\{\deg P(z)Q_{k,0}(z^{2^m}), P_{k,0}(z^{2^m})\} \leqslant 2^m(k+1).$$

Define the integers

$$p_{k,m} := b^{2^m(k+1)} P_{k,m}(1/b)$$

and

$$q_{k,m} := b^{2^m(k+1)} Q_{k,m}(1/b).$$

Since h_k is nonzero there exist positive real constants $c_i(k)$ $(i = 3, \ldots, 6)$ depending only on k so that

$$c_3(k) b^{2^m(k+1)} \leqslant q_{k,m} \leqslant c_4(k) b^{2^m(k+1)}, \tag{13}$$

and

$$\frac{c_5(k)}{b^{2^m(2k+1)}} \leqslant \left|\mathcal{H}(1/b) - \frac{p_{k,m}}{q_{k,m}}\right| \leqslant \frac{c_6(k)}{b^{2^m(2k+1)}}. \tag{14}$$

Thus by (13) there are positive constants $c_7(k)$ and $c_8(k)$ such that

$$\frac{c_7(k)}{q_{k,m}^2} \leqslant \frac{1}{b^{2^{m+1}(2k+1)}} \leqslant \frac{c_8(k)}{q_{k,m}^2}.$$

Applying this to (14) yields

$$\frac{c_9(k)}{q_{k,m}^{1+\frac{k}{k+1}}} \leqslant \left|\mathcal{H}(1/b) - \frac{p_{k,m}}{q_{k,m}}\right| \leqslant \frac{c_{10}(k)}{q_{k,m}^{1+\frac{k}{k+1}}},$$

for some positive constants $c_9(k)$ and $c_{10}(k)$, from which we deduce that

$$\frac{c_9(k)}{q_{k,m}^2} \leqslant \left|\mathcal{H}(1/b) - \frac{p_{k,m}}{q_{k,m}}\right| \leqslant \frac{c_{10}(k)}{q_{k,m}^{1+\frac{k}{k+1}}}. \tag{15}$$

Let $K \geqslant 1$ be an integer and denote by $m_0(k)$ the integer such that for $m \geqslant m_0(k)$ the sequence $\{q_{k,m}\}_{m \geqslant m_0(k)}$ is increasing. We define the sequence of positive integers $\{Q_{K,n}\}_{n\geqslant 1}$ as the sequence of all the integers $q_{k,m}$ with $k+1$ odd, $2^{K-1}+1 \leqslant k+1 \leqslant 2^K+1$, $m \geqslant m_0(k)$, put in increasing order. Then by Lemma 13 and (13) there is an $n_0(K)$ and a positive constant $C_0(K)$ such that

$$Q_{K,n} < Q_{K,n+1} \leqslant C_0(K) Q_{K,n}^{\frac{2^{K-1}+3}{2^{K-1}+1}} \tag{16}$$

for all $n \geqslant n_0(K)$. By (15), there are positive integers $P_{K,n}$ and $Q_{K,n}$ and positive constants $C_1(K)$ and $C_2(K)$ such that

$$\frac{C_1(K)}{Q_{K,n}^2} \leqslant \left|\mathcal{H}(1/b) - \frac{P_{K,n}}{Q_{K,n}}\right| \leqslant \frac{C_2(K)}{Q_{K,n}^{1+\frac{2^K}{2^K+1}}}; \tag{17}$$

here we have taken $P_{K,n}$ to be the $p_{k,m}$ associated to $q_{k,m} = Q_{K,n}$.

Applying Lemma 12, using (17) and (16), we have that

$$\mu\left(\mathcal{H}(1/b)\right) \leqslant 2\left(\frac{2^{K-1}+3}{2^{K-1}+1}\right)\left(\frac{2^K+1}{2^K}\right).$$

Since K can be taken arbitrarily large, we have that $\mu\left(\mathcal{H}(1/b)\right) \leqslant 2$, and since $\mathcal{H}(1/b)$ is transcendental (or even just using irrationality) we have that

$$\mu\left(\mathcal{H}(1/b)\right) = 2.$$

Choosing $\epsilon = -1$ gives that $\mu\left(\mathcal{G}(1/b)\right) = 2$, and that choosing $\epsilon = 1$ gives that $\mu\left(\mathcal{F}(1/b)\right) = 2$. This proves the theorem. $\square$

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References

[APWW] J.-P. Allouche, J. Peyrière, Z.-X. Wen, and Z.-Y. Wen, *Hankel determinants of the Thue-Morse sequence*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 1, 1–27.

[AR] B. Adamczewski and T. Rivoal, *Irrationality measures for some automatic real numbers*, Math. Proc. Cambridge Philos. Soc. **147** (2009), no. 3, 659–678.

[Bre] C. Brezinski, *Padé-type approximation and general orthogonal polynomials*, International Series of Numerical Mathematics, vol. 50, Birkhäuser Verlag, Basel, 1980.

[Bug] Y. Bugeaud, *On the rational approximation of the Thue–Morse–Mahler number*, preprint.

[Coo] M. Coons, *Extension of some theorems of W. Schwarz*, Canad. Math. Bull. (2011), doi:10.4153/CMB–2011–037–9.

[Duv] D. Duverney, *Transcendence of a fast converging series of rational numbers*, Math. Proc. Cambridge Philos. Soc. **130** (2001), no. 2, 193–207.

[Gol] S. W. Golomb, *On the sum of the reciprocals of the Fermat numbers and related irrationalities*, Canad. J. Math. **15** (1963), 475–478.

[Lio] J. Liouville, *Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationelles algébriques*, C.R. Acad. Sci. Paris **18** (1844), 883–885, 910–911.

[Mah1] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. **101** (1929), no. 1, 342–366.

[Mah2] K. Mahler, *Arithmetische Eigenschaften einer Klasse transzendental-transzendenter Funktionen*, Math. Z. **32** (1930), no. 1, 545–585.

[Mah3] K. Mahler, *Uber das Verschwinden von Potenzreihen mehrerer Veränderlichen in speziellen Punktfolgen*, Math. Ann. **103** (1930), no. 1, 573–587.

[Rot] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20; corrigendum, 168.

[Sch] W. Schwarz, *Remarks on the irrationality and transcendence of certain series*, Math. Scand **20** (1967), 269–274.

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